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BRST Algebra Quantum Double and Quantization of the Proper Time Cotanjent Bundle

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Abstract

The quantum double for the quantized $BRST$ superalgebra is studied. The corresponding R-matrix is explicitly constucted. The Hopf algebras of the double form an analytical variety with coordinates described by the canonical deformation parameters. This provides the possibility to construct the nontrivial quantization of the proper time supergroup cotangent bundle. The group-like classical limit for this quantization corresponds to the generic super Lie bialgebra of the double.

1 Introduction

The theories of Casalbuoni-Brink-Schwarz (CBS) superparticle [1] are fundamentally related to supersymmetric field theories and strings. Superparticle orbits are determined up to local fermionic (Siegel) transformations [2] , which play crutial role in removing the unphysical degrees of freedom. For the case of superparticle it has been shown [3] that Siegel symmetry can be interpreted as the usual local proper-time supersymmetry (PTSA). The equivalence between CBS-superparticle and the spinning particle was established [4] by identifying Lorentz-covariant Siegel generator with the local proper-time supersymmetry of the spinning particle [5].

To quantize such models it is natural to apply the BRST formalism, which is manifestly Lorentz-invariant. For the point particle case the BRST quantization starts with the Faddeev-Popov prescription and the extraction of a new nilpotent symmetry operator. The latter can be included in the algebra ILI(1) [6].

Thus the symmetry algebra of a system with superparticles contain both $BRST$ and $PTSA$ subalgebras. The simpliest possible unification of them is the direct sum. It is natural to consider the properties of quantum analogues of $(PTSA) \oplus (BRST)$. On the other hand $BRST$ algebra itself can be treated as a deformation of the trivial algebra of coordinate functions for the superparticle. So one can equally consider q-deformations of a unification of $PTSA$ with Abelian superalgebra creating the $BRST$ subalgebra in the process of deformation. In this case the initial unification is a semidirect sum corresponding to the coadjoint action.

The significant feature of the symmetries $PTSA$ and $BRST$ is that their superalgebras are dual. This gives the opportunity to obtain the necessary q-deformed symmetry by constructing Drinfeld double for a quantized $(PTSA)_q$ superalgebra. The latter is easily obtained using the method developed in [7].

In this paper we demonstrate that the Hoph algebra of the quantum double $SD(PTSA_q, BRST_q)$ can be treated as a quantized symmetry for both interpretation schemes presented above. For the first one the double must be considered as a quantum group corresponding to the algebra $(PTSA) \oplus (BRST)^{\rm opp}$. In the second approach the multiplications in SD are treated as the deformedalgebraof coadjoint extension of $(PTSA)$.

The paper is organized as follows. In the second section all the neces-

sary algebraic constructions are obtained including the explicite expression of $\mathcal{R}$-matrix for $SD(PTAS_q, BRST_q)$. In section 3 the dual canonical parameters are introduced in SD. This gives the possibility to construct the limit transitions connecting different Poisson structures in the created set of Hopf algebras. All the necessary classical limits are explicitely realized. The obtained results are discussed in section 3 from the point of view of possible physical interpretation.

2 The BRST algebra quantum double

Let the Hopf algebra with the generators $\{T, S\}$ and the defining relations

$$\begin{array}{l} [T, S] = 0; \\ \{S, S\} = 2\frac{\sinh(hT)}{\sinh(h)}; \\ \Delta T = T \otimes 1 + 1 \otimes T; \\ \Delta S = e^{hT/2} \otimes S + S \otimes e^{-hT/2}. \end{array} \qquad (1)$$

be interpreted as the proper-time quantum superalgebra ($PTSA_q$). Chose the following quantization of the two-dimensional $BRST$-algebra with basic elements $\{\tau, \xi\}$:

$$\begin{array}{l} [\tau, \xi] = \frac{h}{2}\xi; \\ \{\xi, \xi\} = 0; \\ \Delta\tau = \tau \otimes 1 + 1 \otimes \tau + \frac{h}{\sinh(h)}\xi \otimes \xi; \\ \Delta\xi = \xi \otimes 1 + 1 \otimes \xi. \end{array} \qquad (2)$$

Consider generators τ and ξ as dual to T and S. Then the algebra (2) can be treated as dual opposite to $PTSA_q$, that is the $(PTSA_q)^*$, with opposite comultiplication and inverse antipode.

Note that according to the quantum duality principle [8, 9] the $PTSA_q$ algebra defines also the quantization of the 2-dimensional vector quantum group described by the coproducts in (1). This is the semidirect product of two abelian groups and its supergroup nature is reflected only by the fact that its topological space is a superspace. The quantum supergroup (different from the previous one) is also defined by the Hopf algebra $BRST_q$ (see Δ's in (2)).

To obtain the quantum superdouble $SD(PTSA_q, BRST_q)$ one can start by constructing the corresponding universal element. Let us define the Poincare-Birkhoff-Witt-basis for $PTSA_q$ and $BRST_q$:

$$\begin{array}{l} 1, \xi, \frac{\tau^n}{n!}, \frac{\xi\tau^n}{n!} \\ 1, S, \frac{T^n}{n!}, \frac{ST^n}{n!}. \end{array} \tag{3}$$

The universal element can be written in the form

$$\mathcal{R} = (1 \otimes 1 + S \otimes \xi) e^{T \otimes \tau}. \tag{4}$$

Its main properties are easily checked with the help of an auxiliary relation

$$\begin{array}{c} \left(1 \otimes 1 \otimes 1 + \frac{(e^{2hT}-1)}{e^h - e^{-h}} \otimes \xi \otimes \xi\right) \exp\left(T \otimes 1 \otimes \tau + T \otimes \tau \otimes 1\right) = \\ = \exp\left(T \otimes \tau \otimes 1 + T \otimes 1 \otimes \tau + \frac{h}{\sinh(h)} T \otimes \xi \otimes \xi\right). \end{array} \tag{5}$$

Next step should involve the construction of the multiplication rules consistent with this $\mathcal{R}$-matrix. For any pair of dual Hopf algebras H and H^* with the basic elements $\{e_s\}$ and $\{e^t\}$ and the universal element $\mathcal{R} = e_s \otimes e^s$ the following relation is valid both for ordinary Hopf algebras as well as for super-Hopf ones:

$$\begin{array}{c} (m \otimes \mathrm{id}) \left[(1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{S}^{-1})(\Delta \otimes \mathrm{id})\Delta(e_s)\right] = \\ = (1 \otimes e_s)\mathcal{R}. \end{array} \tag{6}$$

Let us rewrite the third defining relation, $\mathcal{R}\Delta(e) = \tau\Delta(e)\mathcal{R}$, in terms of structure constants,

$$(-1)^{\sigma_k \sigma_l + \sigma_k \sigma_j} \Delta_i^{kl} m_{lj}^t e_k e^j = (-1)^{\sigma_p \sigma_q} \Delta_i^{pl} m_{qp}^t e^q e_l. \tag{7}$$

Here $\sigma_k \equiv \sigma(k)$ is the grading function. From the formulas (6) and (7) the explicit form of multiplication rules follows:

$$e_s e^t = \sum_{n,l,k,u,j} (-1)^{\sigma_n(\sigma_l + \sigma_k) + \sigma_u \sigma_k + \sigma_s \sigma_t} m_{nuk}^t \mu_s^{klj} (\mathbf{S}^{-1})_j^n e^u e_l. \tag{8}$$

Despite the transparency of these rules it is not easy to use them directly. In close analogy with the case of the ordinary double some additional restructuring of the formula (8) is necessary. Calculate two similar expressions: one for the element e^t,

$$\Phi(e^t) \equiv (-1)^{\sigma_u \sigma_k} m_{nuk}^t e^n \otimes e^k \otimes e^u, \tag{9}$$

the other for e_s,

$$\begin{array}{c}\Psi(e_s) \equiv (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \mathrm{id} \otimes \mathbf{S}^{-1})\Box(e_s) = \\ (-1)^{\sigma_l\sigma_j+\sigma_k\sigma_j}\Box_s^{klj}(\mathbf{S}^{-1})_j^n e_n \otimes e_k \otimes e_l,\end{array} \tag{10}$$

with $\Box \equiv \mu(\mu \otimes \mathrm{id})$, μ – the multiplication in the dual Lie superalgebra ($BRST_q$ in our case). To write down the product $e_s \cdot e^t$ it is sufficient to contract the first and the second tensor factors and to multiply the third ones:

$$(-1)^{\sigma_s\sigma_t} e_s \cdot e^t = \left\langle \Phi^{'}(e^t), \Phi^{'}(e_s)\right\rangle \left\langle \Phi^{''}(e^t), \Phi^{''}(e_s)\right\rangle \Phi^{'''}(e^t) \cdot \Phi^{'''}(e_s). \tag{11}$$

Applying these formulas to the pair $(PTSA_q, BRST_q)$ we obtain the Hopf superalgebra $SD(PTSA_q, BRST_q)$ with the defining relations:

$$\begin{array}{ll} [T,S] = 0; & \\ [\tau,\xi] = \frac{h}{2}\xi; & \{S,S\} = 2\frac{\sinh(hT)}{\sinh(h)}; \\ [S,\tau] = hs - 2\frac{h\xi}{\sinh(h)}\cosh(\frac{1}{2}hT); & \{\xi,\xi\} = 0; \\ [T,\tau] = 0; & \{s,\xi\} = 2\sinh\left(\frac{1}{2}hT\right); \\ [T,\xi] = 0; & \end{array} \tag{12}$$

$$\begin{array}{l} \Delta T = T \otimes 1 + 1 \otimes T; \\ \Delta\xi = \xi \otimes 1 + 1 \otimes \xi; \\ \Delta S = e^{\frac{hT}{2}} \otimes S + S \otimes e^{-\frac{hT}{2}}; \\ \Delta\tau = \tau \otimes 1 + 1 \otimes \tau + \frac{h}{\sinh(h)}\xi \otimes \xi; \end{array} \tag{13}$$

$$\begin{array}{ll} \mathbf{S}(T) = -T; & \mathbf{S}(\tau) = -\tau; \\ \mathbf{S}(S) = -S; & \mathbf{S}(\xi) = -\xi. \end{array} \tag{14}$$

It is easy to check that the universal $\mathcal{R}$-matrix (4) realize the triangularity of this quantum superdouble.

3 Deformations of super Lie-Poisson structures induced by superdouble

Applying quantum duality to the algebra $PTSA_q$ one can introduce the canonical parameter p dual to h [10]. The composition

$$\{s,s\} = 2p\frac{\sinh(hT)}{\sinh(h)}$$

is the only relation that changes. In the $(BRST)_q$-algebra the co-product $\Delta(\tau)$ also aquires the dual parameter:

$$\Delta\tau = \tau \otimes 1 + 1 \otimes \tau + \frac{hp}{\sinh(h)} \xi \otimes \xi;$$

(compare with (2)). As a result we obtain the two-parametric family $SD^{hp}(PTSA, BRST)$ of quantum doubles. It can be observed that in the Hopf algebra (12,13,14) the composition $[\tau, \xi]$ allows the rescaling

$$[\tau, \xi] = \frac{1}{2} \alpha h \xi$$

with the additional arbitrary parameter α. We shall consider the case $\alpha = 2$ (in order to have the necessary classical limts) and chose the one-dimensional family of Hoph algebras putting $p = 1 - h$. The defining relations for $SD^{h,1-h}_{\alpha=2} \equiv SD^{(h)}$ are

$$\begin{array}{l} [\tau, \xi] = h\xi; \\ \{S, S\} = 2(1-h) \frac{\sinh(hT)}{\sinh(h)}; \\ \{S, \xi\} = 2 \sinh\left(\frac{hT}{2}\right); \\ [S, \tau] = hS - \frac{2h(1-h)}{\sinh(h)} \xi \cosh\left(\frac{hT}{2}\right); \\ \Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \frac{h(1-h)}{\sinh(h)} \xi \otimes \xi; \\ \Delta(S) = \exp\left(\frac{1}{2} hT\right) \otimes S + S \otimes \exp\left(-\frac{1}{2} hT\right); \end{array} \qquad (15)$$

(from here on we expose only nonzero supercommutators and nonprimitive coproducts).

According to the general theory of quantum double [9] the elements of the set $SD^{(h)}$ can be presented as the deformation quantizations, the corresponding Lie superbialgebra can be constructed using the classical Manin triple. Now we shall show that the set $SD^{(h)}$ induces deformations of super Lie-Poisson (SL-P) structures thus attributed to the Hopf algebras in $SD^{(h)}$.

Consider the Hopf algebra $H^{(0)} \in SD^{(h)}$ described by the relations (15) in the limit $h \to 0$:

$$\begin{array}{l} [S, \tau] = -2\xi; \\ \{S, S\} = 2T; \end{array} \qquad (16)$$

$$\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \xi \otimes \xi. \qquad (17)$$

This limit can be interpreted as a quantized semidirect product $(PTSA \vdash \mathrm{Ab})_q$. The corresponding analytical [11] variety $\mathcal{D}^{(0)}_{\mu\theta}$ of Hopf algebras is defined by the compositions

$$\begin{array}{l} [S,\tau] = -2\mu\xi; \\ \{S,S\} = 2\mu T; \\ \Delta(\tau) = \tau\otimes 1 + 1\otimes\tau + \theta\,\xi\otimes\xi. \end{array} \tag{18}$$

These relations correspond to the quantized SL-P structure in which the cocommutative superalgebra $(PTSA \vdash \mathrm{Ab})$ is deformed in the direction of the Poisson bracket $\{\xi,\xi\} = \tau\theta$. This quantization looks trivial, the multiplications in (18) do not depend on θ.

In the opposite limit $h \to 1$ the Hopf algebra $H^{(1)} \in SD^{(h)}$ presents a nontrivial deformation of a semidirect product $(BRST \vdash \mathrm{Ab})$:

$$\begin{array}{l} [\tau,\xi] = \xi; \\ [S,\tau] = +S; \\ \{S,\xi\} = 2\sinh\left(\frac{T}{2}\right); \end{array} \tag{19}$$

$$\Delta(S) = \exp\left(\frac{1}{2}T\right)\otimes S + S\otimes\exp\left(-\frac{1}{2}T\right). \tag{20}$$

The procedure analogous to that used for $H^{(0)}$ leads to the analytical variety $\mathcal{D}^{(1)}_{\mu\theta}$ of Hopf algebras

$$\begin{array}{l} [S,\tau] = +\mu S; \\ [\tau,\xi] = \mu\xi; \\ \{S,\xi\} = 2\frac{\mu}{\theta}\sinh\left(\frac{\theta T}{2}\right); \\ \Delta(S) = \exp\left(\frac{1}{2}\theta T\right)\otimes S + S\otimes\exp\left(-\frac{1}{2}\theta T\right). \end{array} \tag{21}$$

They have dual classical limits. The two varieties $\mathcal{D}^{(0)}_{\mu\theta}$ and $\mathcal{D}^{(1)}_{\mu\theta}$ intersect in the trivial point – the Abelian and coAbelian Hopf algebra $H^{(0)}_{00} = H^{(1)}_{00}$.

Let us show that there exists the contineous deformation [11] of the SP-L structure $\mathcal{D}^{(0)}_{\mu\theta}$ in the direction of $\mathcal{D}^{(1)}_{\mu\theta}$. The first order deforming functions for such a deformation is a field on $\mathcal{D}^{(0)}_{\mu\theta}$ tangent to the flow connecting $\mathcal{D}^{(0)}_{\mu\theta}$ and $\mathcal{D}^{(1)}_{\mu\theta}$. Evaluating the difference between the compositions (21) and (18)

and comparing it with the curve (15) as a representative of the flow we get the deforming field $\mathcal{F}^{(0)}_{\mu\theta}$:

$$\begin{array}{l} [S,\tau] = +\mu S + 2\mu\xi; \\ [\tau,\xi] = \mu\xi; \\ \{S,S\} = -2\mu T; \\ \{S,\xi\} = \mu T; \\ \Delta\left(S\right) = \frac{1}{2}\theta\, T\wedge S; \\ \Delta(\tau) = -\theta\,\xi\otimes\xi. \end{array} \tag{22}$$

One can integrate the equations

$$\frac{\partial H^{(h)}_{\mu,\theta}}{\partial h}_{|h=0} = \mathcal{F}^{(0)}_{\mu\theta}$$

imposing the boundary conditions $H^{(0)}_{\mu,\theta} \in \mathcal{D}^{(0)}_{\mu\theta}$, $H^{(1)}_{\mu,\theta} \in \mathcal{D}^{(1)}_{\mu\theta}$, and $H^{(h)}_{1,1} = SD^{(h)}$. One of the possible solutions is the 3-dimensional variety $\mathcal{D}^{(h)}_{\mu\theta}$ of Hopf algebras with compositions

$$\begin{array}{l} [S,\tau] = +\mu h S - \mu\frac{2h(1-h)}{\sinh(h)}\xi\cosh\left(\frac{1}{2}h\theta T\right); \\ [\tau,\xi] = \mu h\xi; \\ \{S,S\} = 2\frac{\mu}{\theta}\left(1-h\right)\frac{\sinh(h\theta T)}{\sinh(h)}; \\ \{S,\xi\} = 2\frac{\mu}{\theta}\sinh\left(\frac{1}{2}h\theta T\right); \\ \Delta\left(S\right) = \exp\left(\frac{1}{2}h\theta T\right)\otimes S + S\otimes\exp\left(-\frac{1}{2}h\theta T\right); \\ \Delta(\tau) = \tau\otimes 1 + 1\otimes\tau + \frac{h(1-h)}{\sinh(h)}\theta\,\xi\otimes\xi. \end{array} \tag{23}$$

For each $h' \in [0,1]$ fixed the 2-dimensional subvariety $\mathcal{D}^{(h')}_{\mu\theta}$ defines the SL-P structure:

$$\begin{array}{l} [S,\tau] = +\mu h' S - \mu\frac{2h'(1-h')}{\sinh(h')}\xi; \\ [\tau,\xi] = \mu h'\xi; \\ \{S,S\} = 2\mu\left(1-h'\right)\frac{h'}{\sinh(h')}T; \\ \{S,\xi\} = \mu h' T; \end{array} \tag{24}$$

$$\begin{array}{l} \delta\left(S\right) = \frac{1}{2}h'\theta T\wedge S; \\ \delta(\tau) = \frac{h'(1-h')}{\sinh(h')}\theta\,\xi\otimes\xi; \end{array} \tag{25}$$

described here as a pair of superalgebra (24) and supercoalgebra (25). For $h' \in (0,1)$ these structures are equivalent. But this is not true for the limit

points – $\mathcal{D}^{(0)}_{\mu\theta}$ and $\mathcal{D}^{(1)}_{\mu\theta}$ represent two different contractions of the quantized SL-P structure $\mathcal{D}^{(h')}_{\mu\theta} \mid_{h' \in (0,1)}$. Thus the main statement is proved: the SL-P structure (16,17) ("trivially" quantized as $\mathcal{D}^{(0)}_{\mu\theta}$) can be deformed in the direction of Hopf algebras belonging to $\mathcal{D}^{(1)}_{\mu\theta}$ (that is – by the field $\mathcal{F}^{(0)}_{\mu\theta}$) to obtain the quantization

$$\begin{array}{l} [S,\tau] = +\mu h S - \mu \frac{2h(1-h)}{\sinh(h)} \xi \cosh\left(\frac{1}{2}h^2 T\right); \\ [\tau,\xi] = \mu h \xi; \\ \{S,S\} = 2\frac{\mu}{h}\left(1-h\right) \frac{\sinh(h^2 T)}{\sinh(h)}; \\ \{S,\xi\} = 2\frac{\mu}{h} \sinh\left(\frac{1}{2}h^2 T\right); \\ \Delta\left(S\right) = \exp\left(\frac{1}{2}h^2 T\right) \otimes S + S \otimes \exp\left(-\frac{1}{2}h^2 T\right); \\ \Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \frac{h^2(1-h)}{\sinh(h)}\, \xi \otimes \xi. \end{array} \tag{26}$$

One of the classical limits (for $\mu \to 0$) lay in the facet $\mathcal{D}^{(h)}_{0\theta}$ of classical supergroups (13). Note that depite these properties the Hopf algebra (26) is a quantization of the same super Lie bialgebra as in the trivial canonical quantization of the proper time group cotangent bundle (18). This is easily checked by evaluating the first order terms in the expansion of the compositions (26) with respect to μ and h. This deformation is induced by the quantum superdouble construction.

Earlier (see [11]) it was demonstrated that quantum double could induce even more complicated deformations of L-P structures where the corresponding groups and algebras of observables are not only deformed but also quantized. In the case disscussed above the procedure presented in [11] does not lead to nontrivial results The variety $\mathcal{D}^{(0)}_{\mu\theta}$ lifted in the domain of non(anti)commutative and nonco(anti)commutative Hopf algebras will have edges equivalent to its internal points. This is a consequence of the equivalence of all the Hopf algebras corresponding to the internal points of $\mathcal{D}^{(h)}_{\mu\theta}$.

4 Conclusions

Analyticity plays the important role in the selection of admissable transformations of Poisson structures. Although the SL-P structures corresponding to $\{\mathcal{D}^{(h')}_{\mu\theta} \mid h' \in (0,1)\}$ are equivalent, the contineous "rotation" of $\mathcal{D}^{(h')}_{\mu\theta}$

breaks the analiticity. This is in accordance with the fact that the compositions (24, 25) with different h''s do not form super Lie bialgebra. This effect was first observed in [12] for a nonsuper case.

The deformations $\mathcal{D}^{(0)}_{\mu\theta} \longrightarrow \mathcal{D}^{(h')}_{\mu\theta}$ might be of considerable physical importance. We would like to stress that in these deformations both the supergroup and the Poisson superalgebra of its coordinate functions are deformed simultaneously. Moreover, the process can not be subdevided into successive deformations of group and algebra for the reasons described above. Thus the deformation of the dynamics must be accompanied by the deformation of the geometry. In our particular case the Lie superalgebra of the cotangent bundle $T^*(PTSG)$ can be quantized (retaining the Hopf structure) if the Abelian subalgebra of the cotangent space is simultaneously deformed into the $BRST$-like algebra and one of the canonical classical limits becomes isomorphic to the classical double of $PTSG$ and $BRST$ groups.

It must be mentioned that other methods of unification such as crossproducts or cocyclic cross- and bicrossproducts of Hopf algebras do not lead to nontrivial algebraic constructions in the case of $PTSA_q$ and $BRST_q$.

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